Analysis of the parameter estimate error when algebraic differentiators are used in the presence of disturbances

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Problem statement

- Parameter estimation using measurements corrupted by high frequency disturbances
- Quantification of estimation error

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Proposal:

- Combination of numerical differentiation and time-varying observers

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- Systematic tuning approaches
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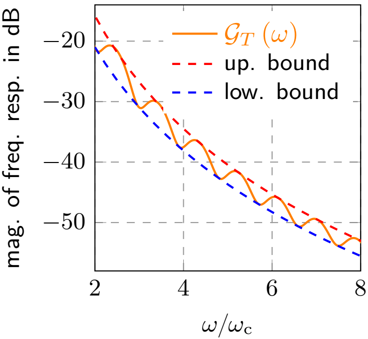
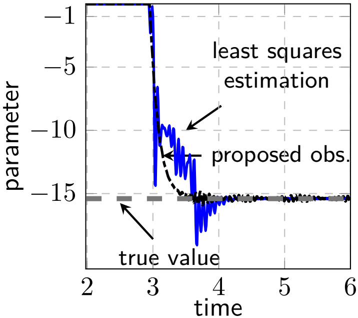
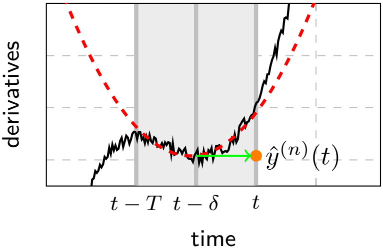




Overview

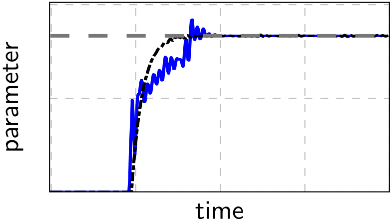
Introduction to algebraic differentiators

- Time-domain analysis
- Frequency-domain analysis



Parameter estimation

- Problem statement
- Identification equation
- Two identification approaches
- Simulation example



Introduction to algebraic differentiators

Approximation[a] of the derivative of a known signal $t \mapsto y(t)$ by

$$\hat{y}^{(n)}(t) = \int_0^T \bar{g}^{(n)}(\tau) y(t-\tau) \mathrm{d}\tau$$

with the filter kernel $\bar{g}$ $\Longrightarrow$ Tuning parameters α, β, N, T, ϑ

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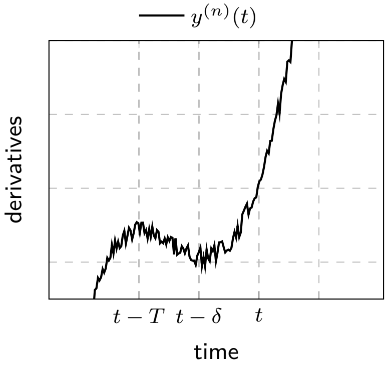

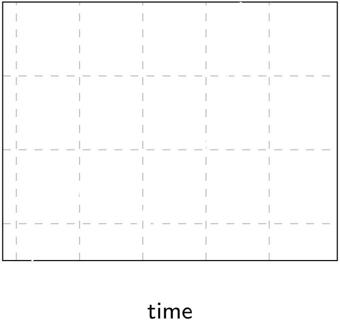

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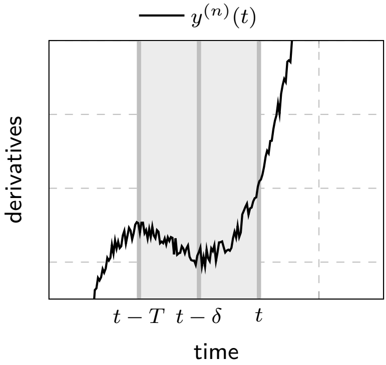

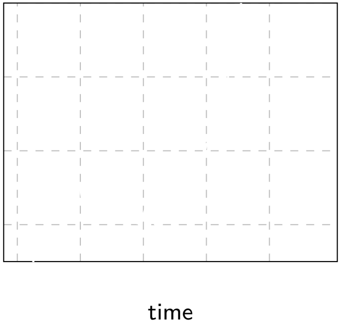

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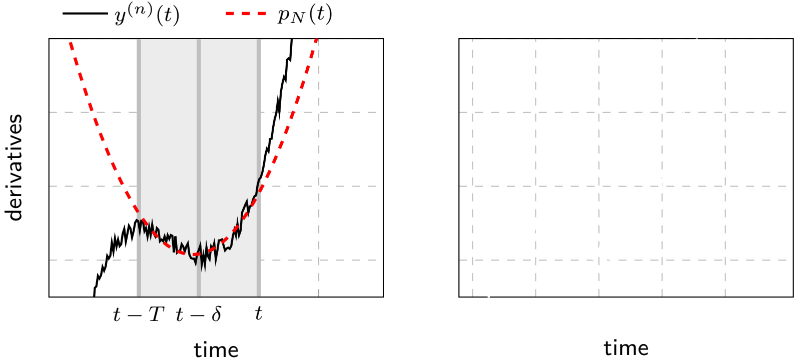

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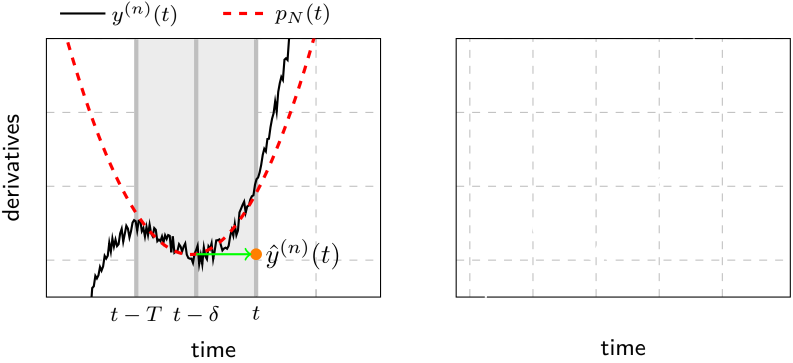

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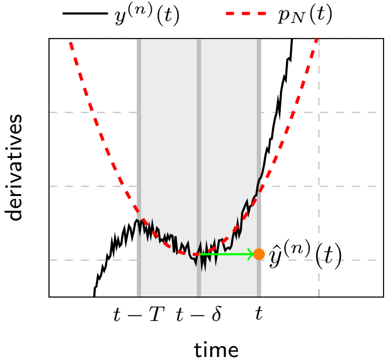

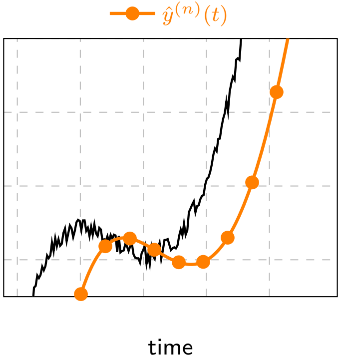

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Estimation errors

$\bar{y}$ a disturbed measurement of a signal y:

$$\bar{y}(t) = y(t) + \eta(t)$$

with η an additive disturbance ⇒ Three sources[a] of errors:

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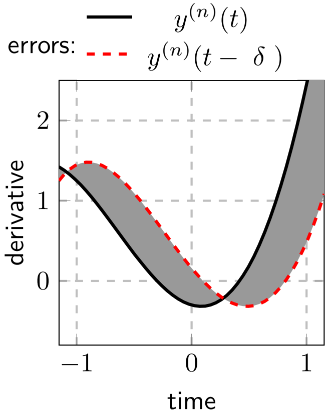
$$e_{\mathsf{d}}(t) := y^{(n)}(t - \delta) - y^{(n)}(t)$$



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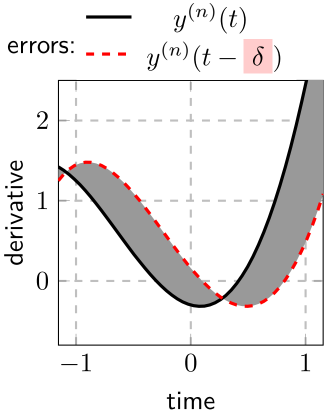

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with η an additive disturbance ⇒ Three sources[a] of errors:

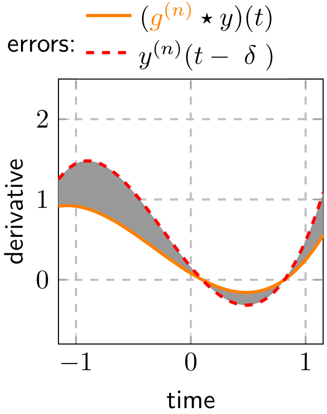
1. Error e_{d} stemming from the delay

2. Error e_{a} from the polynomial approx.:

$$e_{\mathrm{a}}(t) := (g^{(n)} \star y)(t) - y^{(n)}(t - \delta),$$

with $\star$ the convolution operator and

$$g(t) = \begin{cases} \bar{g}(t), & \text{if } t \in [0, T], \\ 0, & \text{otherwise.} \end{cases}$$



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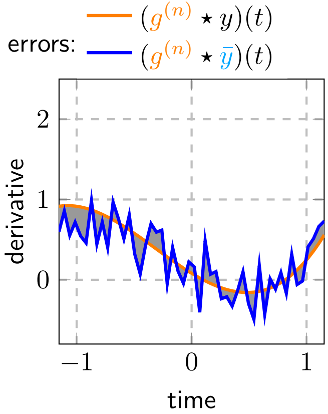

1. Error e_d stemming from the delay

2. Error e_a from the polynomial approx.

3. Error e_n stemming from the disturbance:

$$e_\mathsf{n}(t) := \left(g^{(n)} \star y - g^{(n)} \star \bar{y}\right)(t) = \left(g^{(n)} \star \eta\right)(t)$$

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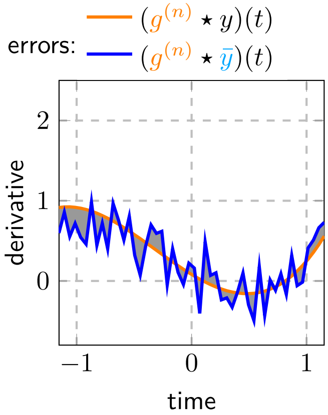

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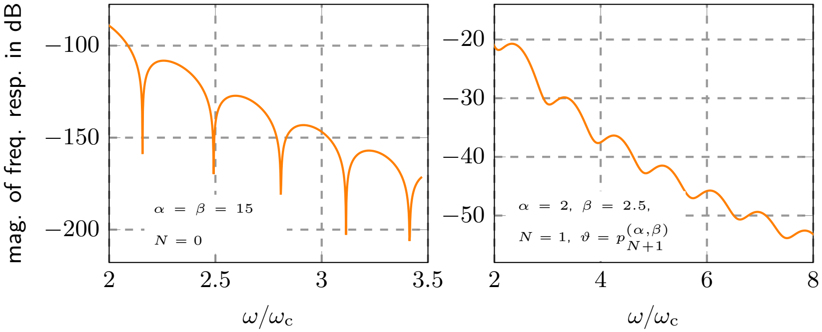
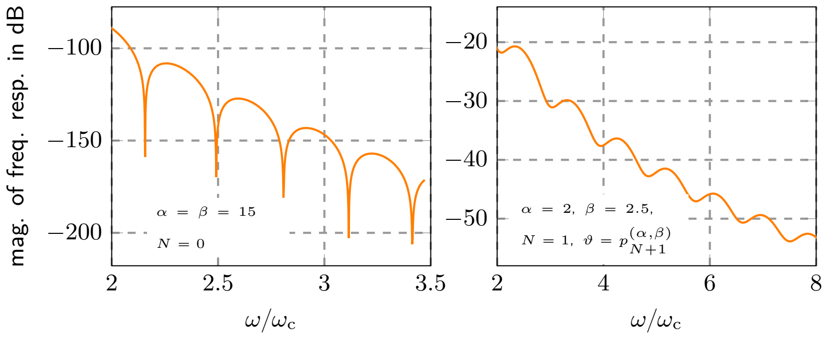
Fourier transform



Fourier transform

$$\mathcal{G}_T(\omega) = \mathrm{e}^{-\iota\omega T} \sum_{i=0}^{N} b_i \sum_{k=0}^{i} c_{i,k} \mathrm{M}_{i,k}^{(\alpha,\beta)}(\iota\omega T), \quad \iota^2 = -1$$

$b_i = \frac{\alpha+\beta+2i+1}{\alpha+\beta+i+1} P_i^{(\alpha,\beta)}(\vartheta)$, $c_{i,k} = (-1)^{i-k}\binom{i}{k}$ and some function $\mathrm{M}_{i,k}^{(\alpha,\beta)}$



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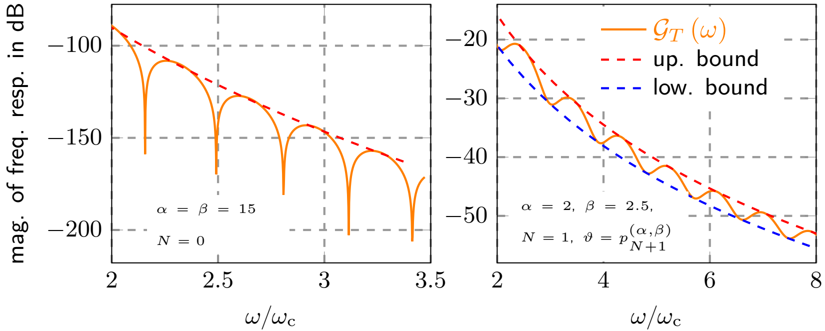
For[b] $|\omega| \to \infty$

$$\frac{1}{(\omega T)^{\mu}}\mathcal{G}^{-} \leqslant |\mathcal{G}_T(\omega)| \leqslant \frac{1}{(\omega T)^{\mu}}\mathcal{G}^{+},\ \mu = \min\{\alpha+1, \beta+1\}$$



[b] L. Kiltz (2017). “Algebraische Ableitungsschätzer in Theorie und Anwendung”. PhD thesis. Universität des Saarlandes

Parameter estimation: Problem statement

- General linear input-output relationship

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i y^{(i)}(t) = b_0 u(t) + \sum_{i=1}^{m} b_i u^{(i)}(t), \quad a_i,\, b_i \in \mathbb{R}.$$

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$$\bar{y}(t) = y(t) + \eta(t)$$

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- Disturbance model

$$\epsilon \dot{w}(t) = S w(t), \quad \eta(t) = P w(t),$$

with

$$S = \text{blockdiag}(S_1, \ldots, S_{\bar{m}}), \quad S_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}, \quad \omega_i \in \mathbb{R}_{>0}$$

$$P = ((0\; p_1), (0\; p_2), \ldots, (0\; p_{\bar{m}})), \quad p_i \in \mathbb{R},\; \bar{m} \in \mathbb{N}$$

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$$P = ((0\ p_1), (0\ p_2), \ldots, (0\ p_{\bar{m}})), \quad p_i \in \mathbb{R},\ \bar{m} \in \mathbb{N}$$

$\Longrightarrow$ Generator of $\bar{m} > 0$ harmonics at frequencies $\frac{w_i}{\epsilon} > 0$, $i = 1, \ldots, \bar{m}$.

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Problem statement: Estimate a_i and b_i and analyze estimation error for small ϵ

Parameter estimation: Identification equation

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$\Longrightarrow y^{(n)}(t) = H_y^T(t)\theta$ with $\theta = [-a_0, \ldots, -a_{n-1}, b_0, \ldots, b_m]^T$

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- Disturbed measurement:

$$\Big(\quad \bar{y}^{(n)}\Big)(t) + a_0\,(\quad \bar{y})\,(t) + \sum_{i=1}^{n-1} a_i \Big(\quad \bar{y}^{(i)}\Big)(t) =$$

$$b_0\,(\quad u)\,(t) + \sum_{i=1}^{m} b_i \Big(\quad u^{(i)}\Big)(t) + (\quad e_\eta)\,(t)$$

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- Disturbed measurement: Apply differentiator on both sides

$$\left(g \quad \star\, \bar{y}^{(n)}\right)(t) + a_0\,(g \star \bar{y})\,(t) + \sum_{i=0}^{n-1} a_i \left(g \quad \star\, \bar{y}^{(i)}\right)(t) =$$

$$b_0\,(g \star u)\,(t) + \sum_{i=0}^{m} b_i \left(g \quad \star\, u^{(i)}\right)(t) + (g \star e_\eta)\,(t)$$

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$$\left(g^{(n)} \star \bar{y} \quad\right)(t) + a_0 \left(g \star \bar{y}\right)(t) + \sum_{i=0}^{n-1} a_i \left(g^{(i)} \star \bar{y} \quad\right)(t) =$$

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$$b_0 \left(g \star u\right)(t) + \sum_{i=0}^{m} b_i \left(g^{(i)} \star u \right)(t) + \left(g \star e_\eta\right)(t)$$

$\Longrightarrow$ $\left(g^{(n)} \star \bar{y}\right)(t) = H_{\bar{y}}^T(t)\theta + \left(g \star e_\eta\right)(t)$: **Only available signals** + error from disturbance

Parameter estimation: Least squares solution

- Let $\lambda > 0$ and

$$\theta_y(t) = \arg\min_{\bar{\theta}} \int_{t_0}^{t} e^{\lambda(\tau - t)} \left(H_y^T(\tau)\bar{\theta} - y^{(n)}(\tau) \right)^2 \mathrm{d}\tau,$$

Assumption $\lim_{t \to \infty} \theta_y(t) = \theta$

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$$\theta_{\bar{y}}(t) = \arg\min_{\bar{\theta}} \int_{t_0}^{t} e^{\lambda(\tau - t)} \left(H_{\bar{y}}^T(\tau)\bar{\theta} - \left(g^{(n)} \star \bar{y} \right)(\tau) \right)^2 \mathrm{d}\tau,$$

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Assumption $\lim_{t\to\infty} \theta_y(t) = \theta$

- Estimation error:

$$\|\theta_y - \theta_{\bar{y}}\|_\infty \leqslant \kappa_g\left(\left(\frac{T}{2}\right)^{p+1}\right) + \kappa_\epsilon\left(\frac{1}{T}, \frac{\omega_>}{\epsilon}\right), \quad \omega_> = \max_{i\in\{1,\ldots,\bar{m}\}} \omega_i$$

with $p \in \mathbb{N}_{>0}$ and κ_g, κ_ϵ class $\mathcal{K}_\infty$ and class $\mathcal{KL}$ functions

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$$\|\theta_y - \theta_{\bar{y}}\|_\infty \leqslant \kappa_g \left(\left(\frac{T}{2} \right)^{p+1} \right) + \kappa_\epsilon \left(\frac{1}{T}, \frac{\omega_>}{\epsilon} \right), \quad \omega_> = \max_{i \in \{1,\ldots,\bar{m}\}} \omega_i$$

with $p \in \mathbb{N}_{>0}$ and κ_g, κ_ϵ class $\mathcal{K}_\infty$ and class $\mathcal{KL}$ functions and

$$\kappa_\epsilon \left(\frac{1}{T}, \frac{\omega_>}{\epsilon} \right) = \mathcal{O} \left(\left| \left(\frac{\omega_>}{\epsilon} \right)^{1-\mu} \right| \right)$$

Parameter estimation: Least squares solution

- Let $\lambda > 0$ and

$$\theta_y(t) = \arg\min_{\bar{\theta}} \int_{t_0}^{t} e^{\lambda(\tau - t)} \left(H_y^T(\tau)\bar{\theta} - y^{(n)}(\tau) \right)^2 \mathrm{d}\tau,$$

$$\theta_{\bar{y}}(t) = \arg\min_{\bar{\theta}} \int_{t_0}^{t} e^{\lambda(\tau - t)} \left(H_{\bar{y}}^T(\tau)\bar{\theta} - \left(g^{(n)} \star \bar{y} \right)(\tau) \right)^2 \mathrm{d}\tau,$$

Assumption $\lim_{t\to\infty} \theta_y(t) = \theta$

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$\Longrightarrow$ Adjust convergence speed with $\mu = \min\{\alpha + 1, \beta + 1\}$.

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- Dynamic solution of Optimization problems $\Longrightarrow$ Online estimation of *slowly varying* parameters:
  - model based control
  - fault detection and localization
  - ...

Observer: delayed outputs

- Dynamic system

$$\dot{\theta}(t) = 0$$
$$y^{(n)}(t) = H_y^T(t)\theta(t)$$

with $\theta(0) = [-a_0, \ldots, -a_{n-1}, b_0, \ldots, b_m]^T$

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- Generating linearly independent signals

$$z(t) = \begin{bmatrix} y^{(n)}(t) \\ \\ \\ \end{bmatrix} = \underbrace{\begin{bmatrix} H_y^T(t) \\ \\ \\ \end{bmatrix}}_{=C(t)} \theta(t),$$

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- Generating linearly independent signals

$$z(t) = \begin{bmatrix} y^{(n)}(t) \\ y^{(n)}(t-\delta_1) \\ \vdots \\ y^{(n)}(t-\delta_\kappa) \end{bmatrix} = \underbrace{\begin{bmatrix} H_y^T(t) \\ H_y^T(t-\delta_1) \\ \vdots \\ H_y^T(t-\delta_\kappa) \end{bmatrix}}_{=C(t)} \theta(t), \ \kappa = n + m$$

with the delays δ_i, $i \in \{1, \ldots, \kappa\}$ such that $\det(C(t)) \not\equiv 0$

Observer: delayed outputs

- Dynamic system

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$$y^{(n)}(t) = H_y^T(t)\theta(t)$$

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- Generating linearly independent signals

$$\hat{z}(t) = \begin{bmatrix} \hat{y}^{(n)}(t) \\ \hat{y}^{(n)}(t-\delta_1) \\ \vdots \\ \hat{y}^{(n)}(t-\delta_\kappa) \end{bmatrix} = \underbrace{\begin{bmatrix} H_{\bar{y}}^T(t) \\ H_{\bar{y}}^T(t-\delta_1) \\ \vdots \\ H_{\bar{y}}^T(t-\delta_\kappa) \end{bmatrix}}_{=\hat{C}(t)} \theta(t),\ \kappa = n + m$$

with the delays δ_i, $i \in \{1, \dots, \kappa\}$ such that $\det\left(\hat{C}(t)\right) \not\equiv 0$ and $\hat{y}^{(i)}(t) = \left(g^{(i)} \star \bar{y}\right)(t)$

Observer: assignable error dynamics

Observer ansatz

$$\dot{\hat{\theta}}(t) = L(t)\left(C(t)\hat{\theta}(t) - z(t)\right), \quad \hat{\theta}(0) \in \mathbb{R}$$

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$$L(t)C(t) = \begin{cases} \left(\det\left(C(t)\right)\right)^2 A, & \text{if } \left|\det\left(C(t)\right)\right| < d \\ \rho A, & \text{otherwise} \end{cases}$$

$d, \rho > 0$ and A Hurwitz

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Lyapunov function $V(t) = e_\theta^T(t)e_\theta(t)$, with $e_\theta(t) = \theta(t) - \hat{\theta}(t)$ $\Longrightarrow$ Exponential convergence of the error

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Lyapunov function $V(t) = e_\theta^T(t) e_\theta(t)$, with $e_\theta(t) = \theta(t) - \hat{\theta}(t)$ $\Longrightarrow$ Bounded error

$+$ bound adjustable via differentiator parameters

Simulation example

Consider the 4-th order linear system defined as

$$\dot{\xi}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \xi(t) + \begin{bmatrix} 0 \\ \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} u(t)$$

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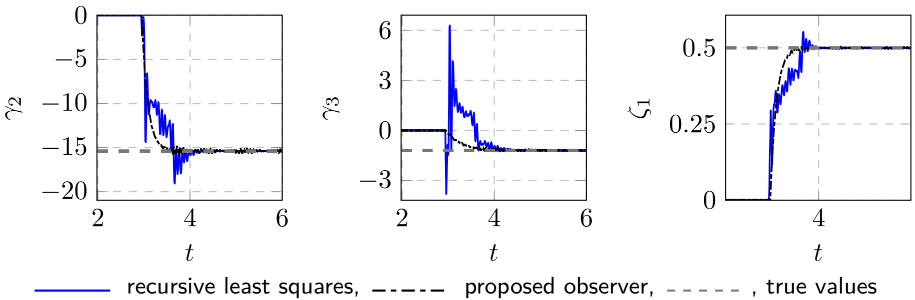
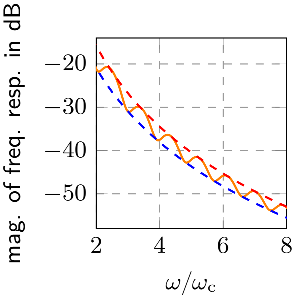


Conclusion

- Exploitation of algebraic differentiators for the estimation of parameters of linear systems



Conclusion

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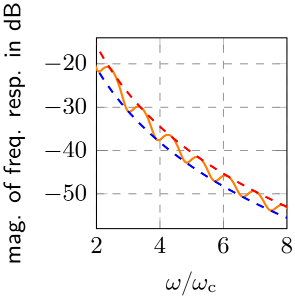
- Derivation of computable error bounds for parameters estimated using algebraic differentiators and a least squares optimization problem



Conclusion

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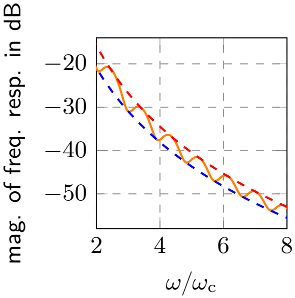
- Presentation of a linear time-varying observer with delayed measurements



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